Lecture 6   02/04/2019

Review of Electrostatics (Con t'd)

Multipole Expansion of the Potential

The electric potential due to a localized charge distribution may be

decomposed into individual contributions from elementary sources,

which is called the <u>multipole expansion</u>. The higher order multipoles

contribute more weakly as the distance from the distribution increases.

Therefore, only the lowest-order non-vanishing multipole is important

at very large distances.

The potential due to the localized

distribution within volume V follows;
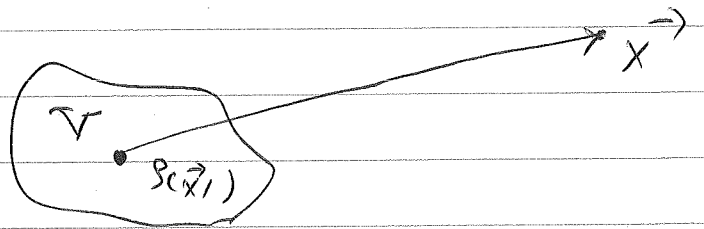
$$\Phi_{(\vec{x})} = \frac{1}{4\pi\epsilon_0} \int_V \frac{S(\vec{x}_1)}{|\vec{x}-\vec{x}_1|} d\sigma'$$

At distances larger than the size of the distribution we have;

$$\frac{1}{|\vec{x}-\vec{x}_1|} = \sum_{\ell,m} \frac{r^\ell}{r^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m} (\theta, \phi) Y_{\ell m}^* (\theta', \phi')$$

Thus:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} q_{lm}$$

Here q_{lm} are the lm-multipole moments of $\rho(\vec{x})$:

$$q_{lm} = \int_V \rho(\vec{x}') r'^l Y_{lm}^*(\theta',\phi') \, d\tau'$$

Let us consider some specific low-order multipoles:

$$q_{00} = \int_V \rho(\vec{x}') Y_{00}^*(\theta',\phi') \, d\tau' = \frac{Q}{\sqrt{4\pi}} \qquad (Q: \text{total charge in } V)$$

$$q_{10} = \int_V \rho(\vec{x}') r' Y_{10}^*(\theta',\phi') \, d\tau' = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}') \underbrace{r' \cos\theta'}_{z'} \, d\tau' = \sqrt{\frac{3}{4\pi}} P_z$$

$$q_{1,1} = \int \rho(\vec{x}') r' Y_{11}^*(\theta',\phi') \, d\tau' = -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{x}') \underbrace{r' \sin\theta' e^{-i\phi'}}_{x'-iy'} \, d\tau' =$$

$$= -\sqrt{\frac{3}{8\pi}} (P_x - iP_y) = -q_{1,-1}^*$$

Here, $\vec{P} = (P_x, P_y, P_z)$, where $\vec{P} = \int_V \rho(\vec{x}') \vec{x}' \, d\tau'$ is the total

electric dipole in V.

$$q_{20} = \int_V \rho(\vec{x}') r'^2 Y_{20}^*(\theta',\phi') \, d\tau' = \sqrt{\frac{5}{4\pi}} \int_V \rho(\vec{x}') \frac{r'^2}{2} (3\cos^2\theta' - 1) \, d\tau'$$

$$= \frac{1}{2} \sqrt{\frac{5}{4\pi}} \int_V \rho(\vec{x}') (3z'^2 - r'^2) \, d\tau' = \frac{1}{2} \sqrt{\frac{5}{4\pi}} Q_{33}$$

Also, $q_{2,2}, q_{2,1}, q_{2,-1}, q_{2,-2}$ can be written as linear combinations of Q_{ij}, where:

$$Q_{ij} = \int_V \rho(\vec{x}')\left(3x'_i x'_j - \delta_{ij} r'^2\right) d\sigma'$$

The Q_{ij}'s are known as the <u>quadrupole moment tensor</u> of the distribution, which is a 3×3 symmetric and traceless matrix.

The usefulness of $q_{\ell m}$'s is that any coordinate rotation mixes those with the same ℓ amongst each other. This is the property of "irreducibility" for each ℓ.

We note that in general the multipole moments $q_{\ell m}$ depend on the choice of the origin. However, the lowest-order non-vanishing multipole(s) do not depend on the choice of origin. This is easily seen in the case of total charge as q_{00} has no dependence on the origin. Also, if $Q = 0$, we have:

$$\vec{P} = \int_V \rho(\vec{x}')\,\vec{x}'\,d\sigma' \xrightarrow[\vec{x}' \to \vec{x}' - \vec{x}_0]{} \int_V \rho(\vec{x}' + \vec{x}_0)\,(\vec{x}' + \vec{x}_0)\,d\sigma' =$$

$$\int_V \delta(\vec{x}')\, \vec{x}'\, d\sigma' + \vec{x}_0 \underbrace{\int_V \delta(\vec{x}')\, d\sigma'}_{Q=0} = \vec{P}$$

The same can be shown for the quadrupole moment if both the total charge and the dipole moment vanish.

An important and useful identity is the following:

$$\int_V \vec{E}(\vec{x})\, d\sigma = \frac{-\vec{P}}{3\varepsilon_0}$$

This holds for a spherical volume V that contains all of the charges, and $\vec{P}$ is the dipole moment due to the charge distribution.

The multipole expansion is also useful for expressing the energy and force of a charge distribution due to an external field. Consider an external electric field $\vec{E}_{ext}$ and a charge distribution $\delta(\vec{x})$ within a volume V in the presence of the external field. The energy of the distribution is:

$$W = \int_V \delta(\vec{x})\, \Phi_{ext}(\vec{x})\, d\sigma$$

Taylor expanding Φ_{ext} about the origin (which is arbitrary) gives:

$$\Phi_{ent}(\vec{x}) = \Phi_{ent}(0) + \vec{x} \cdot \vec{\nabla}\Phi_{ent}(0) + \frac{1}{2}x_i x_j \, \partial_i \partial_j \Phi_{ent}(0)$$

Here, we use the summation convention for repeated indices. Then:

$$W = \Phi_{ent}(0) \int_V \rho(\vec{x}) \, d\tau + \vec{\nabla}\Phi_{ent}(0) \cdot \int_V \rho(\vec{x}) \, \vec{x} \, d\tau + \frac{1}{2}\partial_i \partial_j \Phi_{ent}(0) \int_V \rho(\vec{x}) \, x_i x_j \, d\tau$$

$$+ \cdots$$

Note that:

$$\int_V \rho(\vec{x}) \, d\tau = Q \quad, \quad \int_V \rho(\vec{x}) \cdot \vec{x} \, d\tau = \vec{P} \quad, \quad \vec{\nabla}\Phi_{ent}(0) = -\vec{E}_{ent}(0)$$

Thus:

$$W = Q\,\Phi_{ent}(0) - \vec{P} \cdot \vec{E}_{ent}(0) - \frac{1}{6}\partial_i E_{ent,j}(0) \int_V \rho(\vec{x}) \, (3x_i x_j - \delta_{ij}r^2 + \delta_{ij}r^2) \, d\tau$$

$$+ \cdots = Q\,\Phi_{ent}(0) - \vec{P} \cdot \vec{E}_{ent}(0) - \frac{1}{6}\partial_i E_{ent,j}(0) \int_V \rho(\vec{x}) \, (3x_i x_j - \delta_{ij}r^2) \, d\tau$$

$$+ \frac{1}{6}\partial_i E_{ent,i}(0) \int_V r^2 \, d\tau + \cdots$$

But $\int_V \rho(\vec{x}) \, (3x_i x_j - \delta_{ij}r^2) \, d\tau = Q_{ij}$. Also:

$$\partial_i E_{ent,i}(0) = \vec{\nabla} \cdot \vec{E}_{ent}(0) = 0 \qquad \text{(charges that produce } \vec{E}_{ent} \text{ not contained in } \vec{V})$$

We therefore find:

$$W = Q\,\Phi_{ent}(0) - \vec{P} \cdot \vec{E}_{ent}(0) - \frac{1}{6}\partial_i E_{ent,j}(0) \, Q_{ij} + \cdots$$

The force on the charge distribution is found in a similar way to be:

$$\vec{F} = Q \vec{E}_{ext}(0) + \vec{\nabla}(\vec{p} \cdot \vec{E}_{ext}(0)) + \vec{\nabla}(\frac{1}{6} \partial_i E_{ext,j}(0) Q_{ij}) + \cdots$$

We not that each term in $\vec{F}$ is $-\vec{\nabla}$ of the corresponding term in W, which is expected.